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Dispersal by randomly varying currents

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The long-term oceanic dispersal of persistent contaminants is approached as a problem in turbulent diffusion, with tidal, wind-driven, and other variable currents relegated to turbulence. The mean advection velocity in this problem is typically small compared with the r.m.s. fluctuation. Therefore, close to a continuous concentrated source, puffs of contaminant of all ages are present and have significant effects. Old puffs, i.e. those released a long time previously, give rise to a background concentration field. Young puffs affect the local contaminant concentration p.d.f. according to the probability of their presence, quantified by the visitation frequency.

The behaviour of young puffs is governed by variable advection and may be described approximately in terms of probability distributions obtainable from current-meter data. The visitation frequency can be calculated from the distribution of escape probability density, a Lagrangian equivalent of flux. A long-term effect of variable advection is the distribution of the contaminant over an 'extended' source, which serves as a starting point for the random walk of old puffs. The conventional approach of using the diffusion equation to describe this random walk is therefore valid as a description of the near-source background concentration field, provided that the extended source is used in place of the physical source.

A sample calculation for a typical open coastal case shows that the background concentration plume becomes wide compared with source dimensions, of order K_H/U, with K_H horizontal (eddy and shear) diffusivity, U mean advection velocity. The near-source value of the background concentration is correspondingly low, of order m/hK_n, with m the mass-release rate and h the water depth. Visitation frequencies calculated with the aid of current statistics drop rapidly with distance from the source, especially in the cross-shore direction. The typical cross-shore diameter of the extended source region is a few kilometres.

1. Introduction

Persistent contaminants released into the ocean disperse through advection and mixing in the irregularly varying tidal, wind-driven and thermohaline currents. A plausible point of view is to relegate currents with a timescale of the order of no more than a few days to 'turbulence', and consider a longer-term average (monthly or seasonal) mean flow as steady, the statistical properties of fluctuating currents as stationary. To a so-idealized problem, the classical results of turbulent diffusion theory should apply. However, in this problem the fluctuating velocities are typically much larger (u' of order 30 cm s^{-1}) than the mean (U of order 3 cm s^{-1}), and the standard engineering idealization of a 'slender' plume (Sutton 1953; Brooks 1960; Csanady 1973; Fischer *et al.* 1979) becomes inappropriate. Individual puffs of contaminant are not removed swiftly enough from the source region, so that, in the case of *continuous* release, freshly released puffs may well mingle with puffs released earlier. In the standard plume models it is tacitly supposed that new puffs of contaminant are distributed over a stream of clean fluid. One way to express the difference is to say that the fluid streaming past the source already carries some contaminant, the average or 'background' concentration of which may be significant when the ratio of mean to r.m.s. flow velocity is suitably low.

Other difficulties arise in regard to the question: what is a suitable measure of hazard or nuisance? In the standard models, this is taken to be plume-centre concentration, calculated using relative diffusion theory. It is readily seen, however, that for a large source, and a contaminant decaying slowly, the plume-centre concentration is not very different from concentration immediately following release for many hours – too long for the plume model to remain valid. At target points close to the source, the probability density function (p.d.f.) of the contaminant concentration depends mainly on the frequency of immersion in the plume ('visitation frequency'), with near-background and near-maximum concentrations alternating. Therefore the principal effect of the irregularly varying currents on the concentration p.d.f. at short distances from the source is to cause high or low visitation frequency. Far from the source, on the other hand, any contaminant puffs reaching a target point have wandered irregularly for a long time, and have been subjected to much more vigorous mixing, so that the concentration field is much smoother. Here a fixed-point stochastic mean concentration may be an adequate zeroth-order measure of environmental impact.

One concludes that for low U/u' the standard plume models should be replaced or supplemented by calculations of the visitation frequency and background concentration, as minimal measures of environmental impact. Of course a more detailed knowledge of the concentration p.d.f., or even the joint exposure-time–concentration p.d.f. would also be desirable, but one understandably runs into greater difficulty in attempting to model those.

The modelling problem is approached here on the basis of a distinction between 'young' and 'old' puffs of contaminant. Young puffs advect contaminant in a more or less direct path away from the source, the advection being variable to the extent that velocities occurring at the source are variable. Old puffs, on the other hand, execute a random walk and transport contaminants by a diffusion-like process.

Variable advection of young puffs has much the same effects as the imaginary motion of the source in a direction opposite to that of the instantaneous currents. It results in a distribution of the contaminant over a relatively large mass of fluid, as the flow 'flushes' the source region at the relatively high typical fluctuating velocity. A simple deterministic example is flushing by harmonic tidal currents of amplitudes u_t, v_t: over a full tidal period, contaminant released at a fixed source is distributed over an elliptical area of semi-axes $2u_t/\omega, 2v_t/\omega$, where ω is tidal frequency. Thus the initial effect of variable advection is to *extend* the effective source region to a size comparable to the typical particle excursion during a correlated flow episode.

As already pointed out, young puffs affect the concentration p.d.f. near the source according to their visitation frequency. The distribution of visitation frequency can be approximately calculated from the 'escape probability density', a Lagrangian concept analogous to flux in material diffusion. The concept is latent in some earlier work of Roberts (1961) and Monin & Yaglom (1971). Its explicit use helps also in elucidating the physical processes responsible for the source-extending effect of variable advection.

The contribution of old puffs to the concentration field is given by a solution of the diffusion equation. This is subject to the qualification that a suitably extended source has to be used to parametrize the effect of initial variable advection. It should also be remembered that the solution represents only the 'background' mean concentration field in the vicinity of the source. At large distances from the source, neither of these qualifications affects the result, and standard solutions of the diffusion equation become valid descriptions of the *total* concentration field, because 'young' puffs are not present.

2. Escape probability density

The concept of an escape probability density distribution is simplest in connection with one-dimensional random motion. Consider the displacement $x(t)$ of a particle released at $x = 0$, $t = 0$, and suppose that $x(t)$ is a random function of time. A probability distribution often introduced to describe similar processes is $P(x\,|\,t)$, such that $P(x\,|\,t)\,\mathrm{d}x$ is the probability of a particle being between x and $x+\mathrm{d}x$ at time t (for a three-dimensional version of this function see Batchelor 1949).

A complementary probability distribution arises from the question; how many realizations pass the point x between times t and $t+\mathrm{d}t$? A particle can never be in two places at the same time, but it can be in the same place at two or more times, so that forward and backward crossings of a given abscissa may occur. Correspondingly, forward and backward crossing probabilities may be defined: $Q_+\mathrm{d}t$ and $Q_-\mathrm{d}t$. The difference between these quantities may be termed the probability of escape: with the source at $x = 0$, $Q(x\,|\,t)\,\mathrm{d}t = (Q_+ - Q_-)\,\mathrm{d}t$ is the net probability of a particle moving out of the region $\leqslant x$ surrounding the source and not returning (or, for an assembly of independently moving particles, not being replaced by another particle returning). If the release point of particles is to the right of the point x, $Q(x\,|\,t)$ is negative and is to be interpreted as the probability density of escape in the negative x-direction. The function $Q(x\,|\,t)$, or some analogue of it, is often mentioned in texts dealing with probability theory (Feller 1950; Bartlett 1960). There it is usually called a 'passage-time' distribution.

Generalizing to three dimensions, $P(\boldsymbol{x}, t)$ becomes particle displacement probability per unit volume. Escape probability density is introduced in each of the three coordinate directions, $Q_i(\boldsymbol{x}, t)$ $(i = 1, 2, 3)$ being escape probability per unit time and unit area perpendicular to the x_i axis. Escape probability across any surface element with normal $\boldsymbol{n}$ is readily shown to be

$$Q_n(\boldsymbol{x}, t) = n_i Q_i(\boldsymbol{x}, t), \tag{1}$$

which is the projection of the vector $\boldsymbol{Q}$ with components Q_i onto the normal $\boldsymbol{n}$.

The escape probability density is also readily written down in terms of certain statistical properties of the particle velocity $\boldsymbol{u}$. Let $\phi_c(\boldsymbol{u}\,|\,\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{u}$ be the *conditional* probability that a particle, having reached position $\boldsymbol{x}$ by time t, will have a velocity within element $\mathrm{d}\boldsymbol{u}$ around $\boldsymbol{u}$ in velocity space. A little reflection shows that

$$\begin{aligned} Q_i(\boldsymbol{x}, t) &= \int P(\boldsymbol{x}, t)\, u_i \phi_c(\boldsymbol{u}\,|\,\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{u} \\ &= u_{ti}(\boldsymbol{x}, t)\, P(\boldsymbol{x}, t), \end{aligned} \tag{2}$$

where $u_{ti}(\boldsymbol{x}, t)$ are the first moments of ϕ_c, and may be interpreted as components of a stochastic mean 'transport' velocity.

Permanence of particles is expressed by the conservation law

$$\frac{\partial P}{\partial t}+\frac{\partial Q_i}{\partial x_i}=0. \tag{3}$$

Equivalent expressions have been derived by Roberts (1961) and Monin & Yaglom (1971, equation (9.16)). Neither Roberts nor Monin & Yaglom discuss the physical interpretation of the vector Q_i as a Lagrangian statistical property, independently of any application to a diffusion problem. The above interpretation should be helpful to intuition, especially in the context of the visitation-frequency problem.

When applied to the diffusion of particles a mass M of which have been released at the origin $\boldsymbol{x} = 0$, at time $t = 0$, the stochastic mean concentration χ and flux $\boldsymbol{F}$ are given by

$$\chi = MP(\boldsymbol{x},t), \quad \boldsymbol{F} = M\boldsymbol{Q}(\boldsymbol{x},t). \tag{4}$$

Equation (3) is thus equivalent to an Eulerian mass-conservation law for a diffusing substance. This connection to diffusion theory is useful but incidental, and the results contained in (1)–(3) hold true independently of it.

3. Continuous release

In the case of continuous release, the aggregate effect of all prior particle releases is expressed by integrals over release time t'. Thus the probability of particle presence at $\boldsymbol{x}$ due to all prior releases is, under stationary conditions,

$$p(\boldsymbol{x}) = \int_{-\infty}^{t} P(\boldsymbol{x},t-t')\,\mathrm{d}t' = \int_0^{\infty} P(\boldsymbol{x},t)\,\mathrm{d}t. \tag{5}$$

The physical dimension of this quantity is time per unit volume. It represents an average segment of the time axis, particles originating from which are present simultaneously in the neighbourhood of point $\boldsymbol{x}$, per unit volume.

The integral of the escape probability component $Q_i(\boldsymbol{x}, t-t')$ with respect to release time t' is

$$q_i(\boldsymbol{x}) = \int_{-\infty}^{t} Q_i(\boldsymbol{x},t-t')\,\mathrm{d}t' = \int_0^{\infty} Q_i(\boldsymbol{x},t)\,\mathrm{d}t. \tag{6}$$

This represents the probability of particle escape per unit area, which is now independent of time (under stationary conditions).

Upon integrating the conservation law (3) one arrives at

$$\frac{\partial q_i}{\partial x_i} = \frac{\partial}{\partial x_i}\int_0^{\infty} Q_i(\boldsymbol{x},t)\,\mathrm{d}t = P(\boldsymbol{x},0)-P(\boldsymbol{x},\infty). \tag{7}$$

Even for quite stable substances it is realistic to suppose that some ultimate removal mechanism operates somewhere in the world ocean, so that $P(\boldsymbol{x}, \infty) \approx 0$. Outside source regions therefore

$$\frac{\partial q_i}{\partial x_i} = 0. \tag{8}$$

Integrating around a source on a boundary surface B this may also be written as

$$\int_B n_i q_i \,\mathrm{d}B = 1, \tag{9}$$

where n_i are components of the outward surface normal. A particle released at the source escapes ultimately with probability 1 (analogously to the 'gambler's ruin'; Feller 1950).

In a material-diffusion problem, given continuous release at the origin at the constant rate of m units of mass per unit of time, the stochastic mean concentration and flux are

$$\chi = mp(\boldsymbol{x}), \quad F_i(\boldsymbol{x}) = mq_i(\boldsymbol{x}). \tag{10}$$

Equation (9) expresses the constraint that in a steady-state diffusion problem all of the material released at the origin has to cross the boundary surface B, in the absence of sinks within B. For a variable release rate, $m(t)$ has to be taken inside integral signs defining χ and F_i. In this more general case the results derived below apply with the necessary modifications, which are straightforward.

4. 'Young' and 'old' particles

A kinematic analysis of continuous random particle motion leads to predictions on the asymptotic behaviour of $P(\boldsymbol{x},t)$ at short and long travel times (Taylor 1922; Batchelor 1949, 1952). Although these classical results form the foundations of present-day turbulent diffusion theory, their implications for continuous release, i.e. for the distributions $p(\boldsymbol{x})$ and $q(\boldsymbol{x})$, have not been fully faced. In Gaussian-plume models of air pollution, the initial behaviour of $P(\boldsymbol{x},t)$ can be adequately taken into account (Pasquill 1974), while the behaviour at $t \rightarrow \infty$ is of little concern, because contaminants may be supposed advected rapidly away by the mean flow. In large-scale diffusion problems, on the other hand (see e.g. the study of Spencer, Bacon & Brewer 1981) the initial behaviour of $P(\boldsymbol{x},t)$ in the neighbourhood of any sources is of no interest and can be ignored. Neither of these simplifications is admissible without qualification in the problem of long-term dispersal from a concentrated source, in weak mean flow.

One systematic way to approach the general problem is to separate the contributions to $p(\boldsymbol{x})$ and $q(\boldsymbol{x})$ from 'young' and 'old' particles, i.e. those released a short versus long time ago. Let a dividing time t_d be chosen, and the integral defining $p(\boldsymbol{x})$ split up into two contributions:

$$\left.\begin{aligned} p(\boldsymbol{x}) &= p^{\mathrm{I}}(\boldsymbol{x}) + p^{\mathrm{II}}(x), \\ p^{\mathrm{I}}(\boldsymbol{x}) &= \int_0^{t_\mathrm{d}} P(\boldsymbol{x},t)\,\mathrm{d}t, \\ p^{\mathrm{II}}(\boldsymbol{x}) &= \int_{t_\mathrm{d}}^{\infty} P(\boldsymbol{x},t)\,\mathrm{d}t, \end{aligned}\right\} \tag{11}$$

with a similar decomposition for $\boldsymbol{q}(\boldsymbol{x})$. From (3) one finds now, outside any physical source, i.e. where $P(\boldsymbol{x},0) = 0$,

$$\left.\begin{aligned} \frac{\partial q_i^{\mathrm{I}}}{\partial x_i} &= -P(\boldsymbol{x},t_\mathrm{d}), \\ \frac{\partial q_i^{\mathrm{II}}}{\partial x_i} &= P(\boldsymbol{x},t_\mathrm{d}). \end{aligned}\right\} \tag{12}$$

A physical interpretation is that the young particles move from the physical source region to a virtual sink distribution $P(x,t_\mathrm{d})$, while the old particles originate from a similar source distribution, rather than from the actual, physical source. No approximation has so far been made. However, the point of this decomposition is that the qualitative difference in the behaviour of $P(\boldsymbol{x},t)$ at short versus long travel times may be taken into account by different approximations, which can then be used for the separate estimation of $p^{\mathrm{I}}(\boldsymbol{x})$ and $p^{\mathrm{II}}(\boldsymbol{x})$. The approach is analogous to the modelling of the Lagrangian correlation coefficient R_L by separate assumptions at short and long travel times, and calculating the second moments of dispersion by Taylor's theorem. Because the dispersion is obtained from R_L after integration, the errors in misguessing the shape of R_L at intermediate times are not too serious (see e.g. Sutton 1953; Pasquill 1974). On the other hand, asymptotically incorrect models of R_L lead to serious qualitative errors. Similarly, in the calculations below the use of asymptotic $P(\boldsymbol{x}, t)$ models up to t_d, both from above and below, may well lead to quantitative inaccuracy in the determination of the combined $p(\boldsymbol{x})$ field. However, it avoids the potentially much more serious qualitative errors that result from altogether ignoring the contributions of either the young or the old particles. At the same time, any specific choice of t_d is clearly arbitrary, and different choices lead to different 'splits' between young and old particle fields. The indeterminacy so introduced does not affect the general conclusions one may draw from such calculations. Some examples are given below.

5. Dispersal by random advection

At a particle travel time short compared with the Lagrangian timescale t_L, introduced by Taylor (1922), the shape of the distribution $P(\boldsymbol{x}, t)$ is controlled by direct advection from the source. This is so because at such a short time after release particle velocity is nearly the same as at release. Therefore, in many trials, particles come to be distributed in space immediately surrounding the source according to the frequency of velocities affecting them at the source. Let the (unconditional) p.d.f. of the fluid velocity at the source be $\phi(u)$. Then, the overlapping of the two events, arrival at $\boldsymbol{x}$ at time t, and velocity $\boldsymbol{u} = \boldsymbol{x}/t$ is expressed, in three-dimensional space (see e.g. Monin & Yaglom 1971, equation (9.25)), by

$$P(\boldsymbol{x}, t) = \frac{1}{t^3}\phi\left(\frac{\boldsymbol{x}}{t}\right). \tag{13}$$

The *conditional* probability $\phi_c(\boldsymbol{u})$, introduced in (2), is, on account of the persistence of the velocity:

$$\phi_c(\boldsymbol{u}\,|\,\boldsymbol{x}, t) = \delta\left(\frac{\boldsymbol{x}}{t}\right) \quad (t \ll t_L). \tag{14}$$

The escape probability density is therefore

$$\boldsymbol{Q} = \left(\frac{\boldsymbol{x}}{t}\right) P(\boldsymbol{x}, t) = \left(\frac{\boldsymbol{x}}{t^4}\right)\phi\left(\frac{\boldsymbol{x}}{t}\right) \quad (t \ll t_L). \tag{15}$$

The simplest approximation to $p^{\mathrm{I}}(\boldsymbol{x})$, describing the contribution of random advection to the stochastic mean concentration field, is obtained by choosing t_d of order t_L and substituting (13) into the definition (11). Better approximations can presumably be devised by exploiting the information contained in continuous current-velocity records. The simplest approach yields

$$\left.\begin{aligned} p^{\mathrm{I}}(\boldsymbol{x}) &= \int_0^{t_d} P(\boldsymbol{x}, t)\,\mathrm{d}t = \int_0^{t_d} \frac{1}{t^3}\phi\left(\frac{\boldsymbol{x}}{t}\right)\mathrm{d}t, \\ \boldsymbol{q}^{\mathrm{I}}(\boldsymbol{x}) &= \int_0^{t_d} \frac{\boldsymbol{x}}{t^4}\phi\left(\frac{\boldsymbol{x}}{t}\right)\mathrm{d}t. \end{aligned}\right\} \tag{16}$$

The integration is at fixed $\boldsymbol{x}$, and therefore, in velocity space, along a radius parallel to $\boldsymbol{x}$. This makes it convenient to resolve the velocity p.d.f. $\phi(\boldsymbol{u})$ into a directional frequency $\psi(\beta, \theta)$ and a speed distribution $f(v \mid \beta, \theta)$:

$$\phi(\boldsymbol{u}) = \psi(\beta, \theta) f(v \mid \beta, \theta), \tag{17a}$$

where β, θ are latitude and longitude angles in a spherical coordinate system.

Having in mind applications to the coastal ocean, a two-dimensional version of these results is of greater interest. When the dispersing cloud occupies all available vertical space, between surface and bottom or a thermocline, advection is by the horizontal velocity vector $\boldsymbol{u}$ to a directed horizontal distance $\boldsymbol{x}$, and (13)–(16) apply with the power of t in the denominator reduced by 1. In the frequency distribution only one angle is involved:

$$\phi(\boldsymbol{u}) = \psi(\beta) f(v \mid \beta) \quad (\boldsymbol{u} \text{ two-dimensional}). \tag{17b}$$

Writing $r = |\boldsymbol{x}|$, (16) now become

$$\left.\begin{aligned} p^{\mathrm{I}}(\boldsymbol{x}) &= \frac{\psi(\beta)}{r} \int_{r/t_{\mathrm{d}}}^{\infty} f(v \mid \beta)\, \mathrm{d}v, \\ \boldsymbol{q}^{\mathrm{I}}(\boldsymbol{x}) &= \frac{\boldsymbol{x}}{r^2} \psi(\beta) \int_{r/t_{\mathrm{d}}}^{\infty} v f(v \mid \beta)\, \mathrm{d}v. \end{aligned}\right\} \tag{18}$$

Taking $\psi(\beta)$ to be non-dimensional (probability per unit angle), $f(v|\beta)$ has the physical dimension of v^{-2}, while $p(\boldsymbol{x})$ is now time per unit area of the (x, y)-plane, and $\boldsymbol{q}(\boldsymbol{x})$ escape probability per unit length of a perpendicular line. The ratio of the first and zeroth moments of $f(v \mid \beta)$ serves as a convenient measure of the typical current velocity magnitude in a given direction:

$$v_{\mathrm{f}}(\beta) = \frac{\int_0^{\infty} v f(v \mid \beta)\, \mathrm{d}v}{\int_0^{\infty} f(v \mid \beta)\, \mathrm{d}v}. \tag{19}$$

Very close to the source, $r/t_{\mathrm{d}} \rightarrow 0$, one then has

$$\left.\begin{aligned} p^{\mathrm{I}}(\boldsymbol{x}) &= \frac{\psi(\beta)}{r v_{\mathrm{f}}}, \quad \left(\frac{r}{t_{\mathrm{d}}} \rightarrow 0\right). \\ \boldsymbol{q}^{\mathrm{I}}(\boldsymbol{x}) &= \frac{\boldsymbol{x}}{r^2} \psi(\beta) \end{aligned}\right\} \tag{20}$$

These results are readily interpreted physically. As pointed out before, $p(\boldsymbol{x})$ represents the average time spent by a particle per unit area in the neighbourhood of a given point $\boldsymbol{x}$, in this case around the origin. Taking a small circle of radius r around the origin, the (young) particle stays within this circle according to (20), for a period of order r/v_{f}. In other words, v_{f} is an average velocity at which the particle, released at the source, is 'flushed' away in a given direction β.

At larger distances r, the integration limits in (18) gradually contract, expressing the fact that slow particles get only so far within period t_{d}. Because current speed is bounded, p^{I} tends to zero as $r \rightarrow \infty$: young particles only reach a limited neighbourhood of the source, becoming old before getting very far. This has already been seen in connection with (12).

6. Dispersal by random walk

At long travel times $t \gg t_L$, particle velocity is no longer correlated with the velocity at release, and particle motion takes on the character of random walk. This has often been interpreted as justifying the use of the diffusion equation as a model of turbulent diffusion. Rigorous examination reveals some weaknesses in this argument (Monin & Yaglom 1971) but it also confirms that, for diffusion times greatly in excess of t_L, and regarded as valid only in timesteps of similar length, the diffusion equation should give a reasonable picture of the time evolution of $P(\boldsymbol{x}, t)$. In the absence of distributed particle sources or sinks, this equation is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial P}{\partial x_j} - u_i P\right) \quad (t \gg t_L), \tag{21}$$

where u_i are components of a local mean Lagrangian or mass-transport velocity and K_{ij} those of an eddy-diffusivity tensor. It is important to note that, because one interprets the diffusion equation as applying to changes in time long compared with t_L, u_i and K_{ij} are asymptotic quantities, as one might deduce from a diffusion experiment with a source placed at a given point $\boldsymbol{x}$ in the flow, continuing for a period long compared with t_L. The u_i and K_{ij} are thus still functions of location $\boldsymbol{x}$, but they are smoothed by the imaginary diffusion experiments carried out at neighbouring locations for long periods.

The principal advantage of using the diffusion equation is that nearby or distant sinks, which must ultimately control concentration levels in the case of continuous release, are readily represented, either as distributed sinks (by adding a sink term to (21)) or as fully or partially absorbing boundaries.

Supposing that the diffusion equation applies to the spread of old particles, substitution of (11) into (21) yields

$$\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial p^{\mathrm{II}}}{\partial x_j} - u_i p^{\mathrm{II}}\right) = -P(\boldsymbol{x}, t_d). \tag{22}$$

This is a steady-state form of the diffusion equation, with $P(\boldsymbol{x}, t_d)$ playing the role of a distributed source term, as has already been seen prior to approximating q_i, (12). From classical results in turbulent-diffusion theory one knows that the spatial scale of $P(\boldsymbol{x}, t_d)$ is of order $u't_L$, where u' is a typical fluctuating velocity. At distances large compared with this scale, the solution of (22) is indistinguishable from the field of a point source at the origin. At such distances, furthermore, the contribution of young particles to the combined $p(\boldsymbol{x})$ field vanishes. Hence, far enough from the source, a conventional solution of (22), with the source as specified by the physical conditions of the problem, gives an acceptable representation of the total $p(\boldsymbol{x})$ field, i.e. of the stochastic mean-concentration distribution.

Close to the source the contribution of old particles to the stochastic mean concentration is legitimately interpreted physically as the 'background' concentration, thus giving some precision to a rather vague intuitive notion. It should be clear that this background value depends on how efficiently the diffusion process transports contaminants from the source to some ultimate oceanic sink.

7. Visitation frequency

Close to a source, puffs of both young and old particles are present and contribute to the observable concentration history at a fixed target point. Because contaminant plumes are generally slender, a high degree of intermittency characterizes contaminant concentration within the range of direct excursions from the source (say, those completed in the course of a tidal cycle or a wind-driven flow episode). When the target point is immersed in the plume, a high concentration is observed, otherwise a much lower concentration (of the order of the background concentration). Thus the concentration p.d.f. consists of two widely isolated peaks, a circumstance that makes the total stochastic mean concentration a poor measure of hazard or nuisance. The r.m.s. concentration fluctuation is not much better. A plausible, readily appreciated measure of the intermittency of the concentration is, however, the probability of immersion in a puff of 'young' particles, a quantity to be designated here 'visitation frequency' $\gamma^{\mathrm{I}}(\boldsymbol{x})$.

The order of magnitude of the visitation frequency may be obtained as follows. Consider again the two-dimensional case and suppose that slender plumes emanate from a source region of typical diameter b. Given a moderately large source, the width w of the plumes grows slowly, and, within the range of more or less straight source–target-point excursions, remains close to b. If the plumes were dispersing isotropically, at a distance r they would visit a given location with probability approximately equal to

$$\gamma(r) = \frac{b}{2\pi r}. \tag{23}$$

This is a small quantity even at moderately large r/b. Of course plume dispersion is not isotropic and direct plume excursions do not reach distant target points, so that (23) is very crude.

To arrive at a better estimate, consider puffs of particles simultaneously released at the source. The relative diffusion of these particles is most conveniently referred to the puff's centre of gravity. However, for the present purpose, it is simplest to consider a 'distinguished' particle in the puff, and diffusion relative to that particle. As Batchelor (1952) has shown, the two choices of reference frames are statistically equivalent. In the following, 'puff centre' motion or displacement will be understood as that of the distinguished particle. This is characterized by displacement probability density, $P(\boldsymbol{x}, t)$, and other distributions introduced earlier.

A realized puff trajectory results in observable tracer concentration at an arbitrary target point T if the closest approach distance is equal to puff size or less. To calculate the probability of such an event, let a plane surface be laid through T, perpendicular to puff trajectory at the point A of closest approach (figure 1), containing the approach vector $\boldsymbol{a} = \boldsymbol{TA}$. For the purpose of forming statistics of puff-centre passage, consider all planes normal to puff trajectory at the approach point turned into a standard position (e.g. normal to the radius vector $\boldsymbol{x}$, through the target point T) by pivoting about the intersection of the actual and standard planes. The density of realized vectors $\boldsymbol{a}$ in the standard plane is a measure of the probability that a puff centre of given age $t-t'$ passes within the area element $\mathrm{d}\boldsymbol{a}$ around the point $\boldsymbol{x}+\boldsymbol{a}$.

Let the probability of puff-centre passage through these normal planes per unit area, and per unit time (of puff age) be denoted by $\Pi(\boldsymbol{x}+\boldsymbol{a}, t-t')$. This may be expressed on the model of (2) as

$$\begin{aligned}\Pi(\boldsymbol{x}+\boldsymbol{a}, t-t') &= P(\boldsymbol{x}+\boldsymbol{a}, t-t') \int |\boldsymbol{u}| \phi_{\mathrm{c}}(\boldsymbol{u}|\boldsymbol{x}+\boldsymbol{a}, t-t')\,\mathrm{d}\boldsymbol{u} \\ &= u_{\mathrm{c}}(\boldsymbol{x}+\boldsymbol{a}, t-t')\, P(\boldsymbol{x}+\boldsymbol{a}, t-t'),\end{aligned} \tag{24}$$

where u_{c} is a mean puff-centre velocity magnitude at $\boldsymbol{x}+\boldsymbol{a}$, for puffs of age $t-t'$. Of greatest interest is this passage frequency for young puffs, i.e. at suitably short $t-t'$,

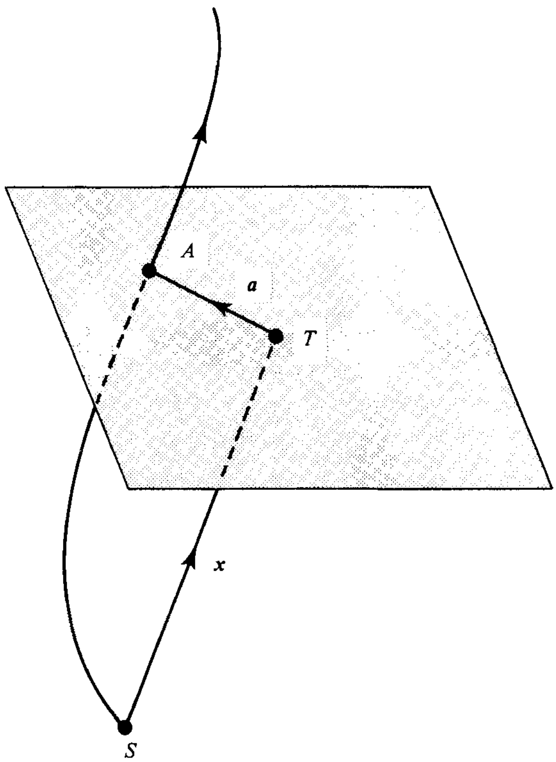

FIGURE 1. Puff-centre trajectory SA approaches target point T to within vector $\boldsymbol{a}$, contained in a plane normal to the trajectory at approach point A.

and close to the source where $u_c \approx |\boldsymbol{u}_t|$, the mean transport velocity magnitude. This is so because velocities at times of (young) puff passage are directed (nearly) along the source–target-point vector. Thus for suitably short $t-t'$ and close to the source,

$$\Pi(\boldsymbol{x}, t-t') \approx |\boldsymbol{Q}(\boldsymbol{x}, t-t')| \quad (\boldsymbol{x}, t-t' \text{ suitably small}) \tag{25}$$

Turning now again to two-dimensional applications, let such young puffs have a width $w(t-t')$. The visitation frequency integrated for puff ages zero to t_d is then

$$\gamma^{\mathrm{I}}(\boldsymbol{x}) = \int_0^{t_d} \int_w |Q(\boldsymbol{x}, t)|\, \mathrm{d}\boldsymbol{a}\, \mathrm{d}t = w_m |\boldsymbol{q}^{\mathrm{I}}(\boldsymbol{x})|, \tag{26}$$

where w_m is a mean width for young puffs. Given a large source, w_m differs little from the projected diameter of the physical source, because relative diffusion is slow at short lifetimes. The distribution of the visitation frequency due to young puffs, at short distances from the source is then readily estimated with the aid of (18) above.

8. Background concentration in coastal waters

According to the discussion above, the contribution of old puffs to the mean concentration field close to a source, legitimately called background concentration, can be modelled as a solution of the diffusion equation. Along many an open coast the depth of the water only varies slowly beyond a kilometre or two from the coast, out to the edge of the continental shelf, taken to be both wide and long. In such an environment a realistic idealization is to suppose constant depth, a constant long-term mean Lagrangian velocity U in the alongshore direction, and a constant horizontal diffusivity K_{H}. Initial mixing (caused by jet momentum and buoyancy) is supposed to distribute the containment over the available depth h, and over some source area of typical radius R. Subsequent advection and diffusion is confined to two horizontal dimensions in an effectively semi-infinite field. These idealizations are realistic, e.g. for the nearshore discharge of the waste heat of a large power station, or for that of domestic sewage or sludge, over a broad, open continental shelf. The Lagrangian alongshore mean current speed U is typically 0.03–0.1 m s^{-1}, and the r.m.s. fluctuating velocity 0.1–0.3 m s^{-1}. The typical persistence time of wind-driven currents is of the order of a day (10^5 s) so that one would expect the effective diffusivity to be of order $\overline{u'^2}\, t_{\mathrm{L}} \approx 10^3$–$10^4$ m^2 s^{-1}. This estimate proves too high: fresh water entering from land diffuses across an open, 'Atlantic'-type continental shelf with a typical diffusivity of $K_{\mathrm{H}} = 300$ m^2 s^{-1} (Ketchum & Keene 1955). The discrepancy is presumably due to the fact that cross-shore velocities are considerably smaller than alongshore ones. In any case, the above value of K_{H} inferred from observation is used in the calculations below. Typical physical source radii may be taken to be 100–1000 m.

Close to a source, where one may think of the solution of the diffusion equation as the background concentration, diffusion along the weak mean current cannot be ignored. The steady-state advection–diffusion equation in two dimensions is

$$Uh\frac{\partial \chi}{\partial x} = K_{\mathrm{H}} h \nabla_1^2 \chi + s(x,y), \tag{27}$$

where $s(x,y)$ is source intensity, or mass released per unit area per unit time. The total source strength is

$$m = \int_{-\infty}^{\infty}\int s(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{28}$$

Non-dimensional variables are conveniently defined as follows:

$$\chi^* = \frac{K_{\mathrm{H}} h \chi}{m}, \quad x^* = \frac{Ux}{2K_{\mathrm{H}}}, \quad y^* = \frac{Uy}{2K_{\mathrm{H}}}, \quad s^* = \frac{4s(x,y)\,K_{\mathrm{H}}^2}{mU^2}. \tag{29}$$

Dropping the asterisks on the non-dimensional variables, (27) becomes

$$2\frac{\partial \chi}{\partial x} = \nabla_1^2 \chi + s(x,y). \tag{30}$$

The solution for a point source $s = \delta(x,y)$ in an infinite field is (Carslaw & Jaeger 1959, p. 267)

$$\chi = \frac{\mathrm{e}^x}{2\pi} K_0(r), \quad r = (x^2+y^2)^{\frac{1}{2}}, \tag{31}$$

with K_0 the modified Bessel function. For a distributed source, the corresponding result is

$$\chi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int s(x',y')\,\mathrm{e}^{x-x'} K_0[(x-x')^2+(y-y')^2]^{\frac{1}{2}}\,\mathrm{d}x'\,\mathrm{d}y'. \tag{32}$$

This is readily evaluated for arbitrary $s(x,y)$, using in the process the asymptotic expression of $K_0(r)$ at small r:

$$K_0(r) = -(\ln \tfrac{1}{2}r + \gamma), \tag{33}$$

where $\gamma = 0.5772$ is Euler's constant. In the present application the above expression may be simplified because the typical source dimension is small:

$$\frac{UR}{2K_{\mathrm{H}}} = 0.025 \ll 1$$

for $R = 0.5$ km, $U = 0.03$ m s^{-1}, $K_{\mathrm{H}} = 300$ m^2 s^{-1}.

The centre value of the concentration is then, for an axisymmetric source $s(x,y) = s(r)$,

$$\chi(0) = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} s(r)\,\mathrm{e}^{-r\cos\phi} K_0(r)\, r\,\mathrm{d}\phi\,\mathrm{d}r$$

$$= \int_0^\infty s(r)\, I_0(r)\, K_0(r)\, r\,\mathrm{d}r$$

$$\approx \int_0^\infty s(r)\,(0.116 - \ln r)\, r\,\mathrm{d}r. \tag{34}$$

The source intensity of the physical source may be supposed constant, i.e.

$$s(r) = \frac{1}{\pi R^2} \quad (r \leqslant R,\ \text{physical source}). \tag{35}$$

The extended source will be modelled by a Gaussian distribution (justified below)

$$s(r) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right) \quad \text{(extended source)}. \tag{36}$$

Substitution into the above result for $\chi(0)$ yields, with the physical source,

$$\chi(0) \approx \frac{1}{2\pi}(1 - \ln R + 0.616)$$

$$= 0.685 \quad \text{(physical source)}, \tag{37}$$

taking the typical non-dimensional $R = 0.025$ quoted above. With the extended source the centre concentration becomes

$$\chi(0) \approx \frac{1}{4\pi}(0.116 - 2\ln\sigma) = 0.311 \quad \text{(extended source)}, \tag{38}$$

having used $\sigma = 3$ km, or a non-dimensional value of 0.15.

The source-extending effect of variable near-source advection is thus responsible for a decrease in background concentration by a factor of 2. This result is, moreover, insensitive to an inaccurate estimation of σ, the variation of $\chi(0)$ with σ being logarithmic.

Numerical integration of (32) yields the details of the concentration field, shown here in figures 2–4. With the physical source, a modest peak near the origin develops (figures 2 and 3). With the extended source the peak is flatter (figure 4). Note, however, that in both cases significant concentrations ($\chi > 0.05$) extend to non-dimensional distances, upstream and cross-stream, of order unity, which in dimensional terms is of order $2K_{\mathrm{H}}/U$ or typically 20 km.

The model discussed here is clearly a steady plume in a slow mean current, and shares some of the shortcomings of standard plume models. The ultimate sink for the contaminant is tacitly supposed to be the deep ocean, near-source concentration being kept within bounds by the flushing action of the mean current. Thus the concentration tends to infinity as the mean current velocity tends to zero (with $-\ln U$). For very

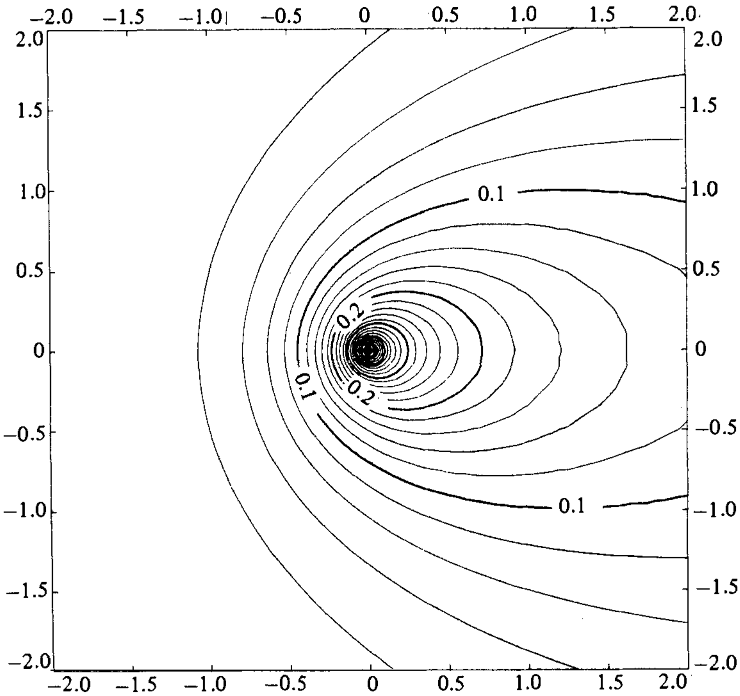

FIGURE 2. Contours of constant concentration around constant-intensity circular source of non-dimensional radius 0.025. Non-dimensional variables used are defined in (29).

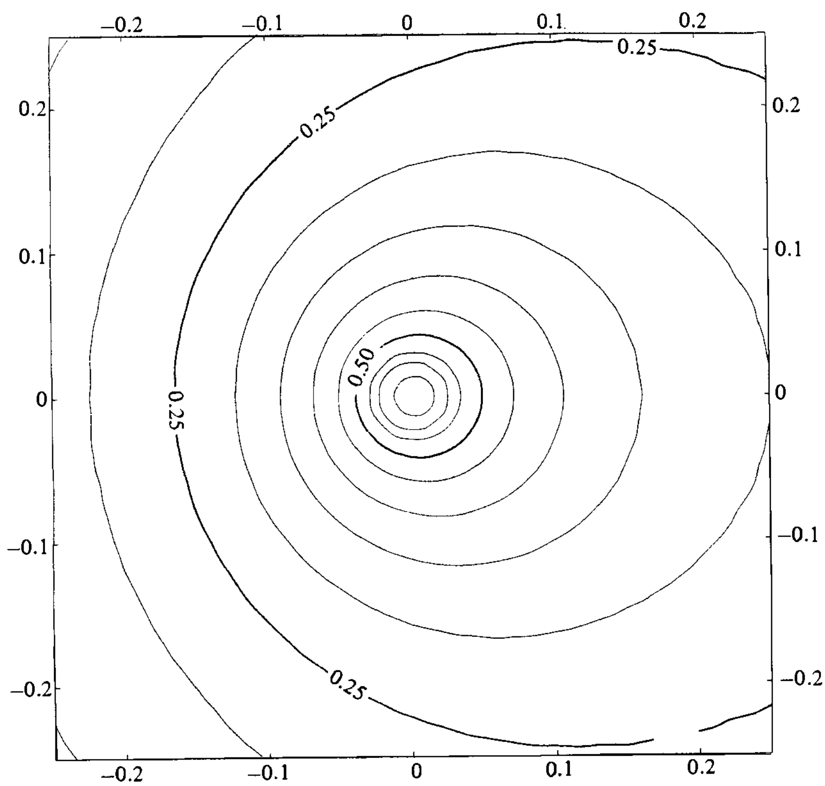

FIGURE 3. As figure 2, showing detail near the source.

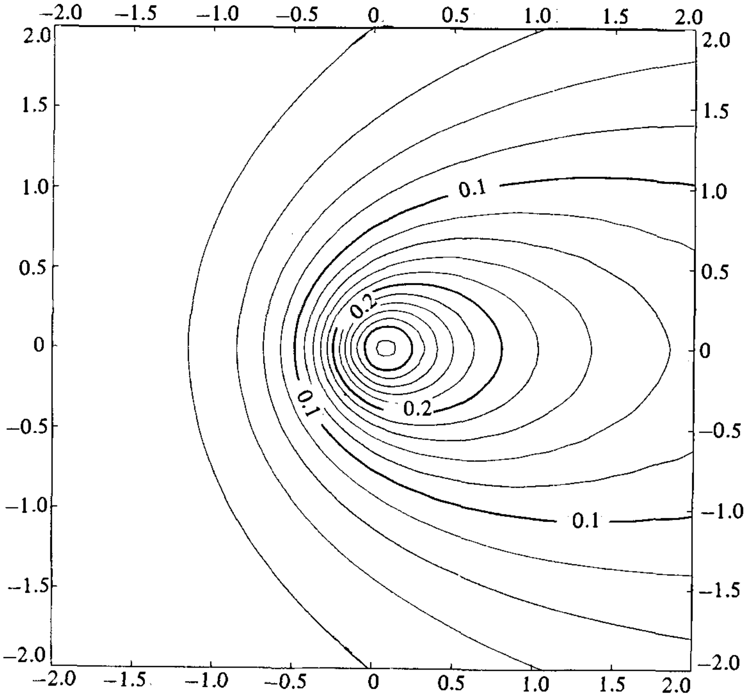

FIGURE 4. Contours of constant concentration around extended source, of Gaussian intensity distribution, with non-dimensional standard deviation 0.15.

low U the result is presumably unrealistic, and the distribution of oceanic contaminant sinks has to be examined more carefully.

To discuss the integrated mass balance further, consider a contour C completely enclosing the source region. An integration of the advection–diffusion equation over the area enclosed by C yields

$$m = h\oint_C \left(U\chi\,\mathrm{d}y - K_{\mathrm{H}}\frac{\partial\chi}{\partial n}\mathrm{d}c\right), \tag{39}$$

where $\partial/\partial n$ denotes differentiation along the outward normal, and $\mathrm{d}c$ is an element of C. If C is taken far enough from the source, and if it contains a straight-line segment parallel to the y-axis, at some large downstream distance x_{c} from the source, the last equation reduces approximately to

$$m \approx Uh\int_{-\infty}^{\infty} \chi(x_{\mathrm{c}})\,\mathrm{d}y = Uh\,\chi_{\mathrm{a}}(x_{\mathrm{c}})\,Y, \tag{40}$$

where $\chi_{\mathrm{a}}(x_{\mathrm{c}})$ is the axial (maximum) concentration at x_{c}, and Y is an effective plume width, defined by this expression. In slender-plume models this approximate mass balance is taken to apply at all distances from the source, and close to the source Y is identified with source diameter. The simplification of (38) into (40) holds, however, only where the Lewis number is large:

$$\frac{UL}{K_{\mathrm{H}}} \gg 1, \tag{41}$$

where $L = \chi(\partial\chi/\partial n)^{-1}$ is the scale of concentration gradients. Close to the source this is not true, and the identification of Y with source diameter cannot, in general, be made. The present example shows that Y is of order $2K_H/U$, which in the case considered is large even compared with the extended source diameter. The result demonstrates that the axial concentration is as small as it is because near-source diffusion spreads the contaminant out over a cross-stream distance of order $2K_H/U$.

Given the large dimensions of the background concentration plume, in any application it is necessary to take into account the presence of a shore at a distance y_S from the source. This is readily accomplished by the well-known image method, the shore being realistically modelled as a reflecting boundary. With the source close to the shore (at $y_S \ll 2K_H/U$) the concentrations double everywhere. The general case is treated equally easily, by a superposition of two concentration fields with offset sources.

The net result is that the non-dimensional background concentration at the source is of order unity, i.e. that the dimensional value χ is

$$\chi \approx \frac{m}{hK_H}. \tag{42}$$

This is as if the contaminant release were distributed over a stream of volume transport hK_H [m^3 s^{-1}]. For $h = 20$ m, $K_H = 300$ m^2 s^{-1} this is a very large 'flushing' volume transport, of order 6000 m^3 s^{-1}. Again a comparison with standard plume models suggests itself: taking the projected area of either the physical or the extended source, a very much smaller flushing volume transport is calculated, resulting in a fictitious 'initial' concentration much higher than has been found here.

As a concrete example, take the case of waste-heat release at the rate of $Q_H = 1000$ MW (10^9 W). In 'kinematic' units this is

$$\frac{Q_H}{\rho c_p} \approx 250 \text{ m}^3 \text{ K s}^{-1}.$$

Distributed over the flushing volume transport of $hK_H = 6000$ m^3 s^{-1}, this causes a background temperature rise of only 0.04 K, without even taking into account heat transfer to the atmosphere. Contrary results derived from a misapplication of standard plume models, quoted in public discussion over power-plant siting, have caused considerable mischief.

9. Near-source effects: estimation from current-meter data

The contribution of young puffs to contaminant concentrations observable at fixed near-source locations is additive to the background field above calculated, but significant only in a limited neighbourhood of the source. Its calculation to lowest order can be carried out simply by postulating persistence of velocity observed at the source, as discussed above. It will be supposed here that a long time series of current-meter observations is available, and that this has been processed to estimate the frequency distribution of currents at the site $\phi(\boldsymbol{u})$.

Let $\phi(\boldsymbol{u})$ be given in sectors $\Delta\beta$ of the current rose, each centred on direction β, and in speed classes Δv, centred on v. Let the total frequency in a given sector be f_β, and the speed-class frequency within the same sector $f_{v\beta}$. Also, introduce the cumulative frequency of currents faster than some speed v, in a given sector β:

$$F_\beta(v) = \sum_v^\infty f_{v\beta}. \tag{43}$$

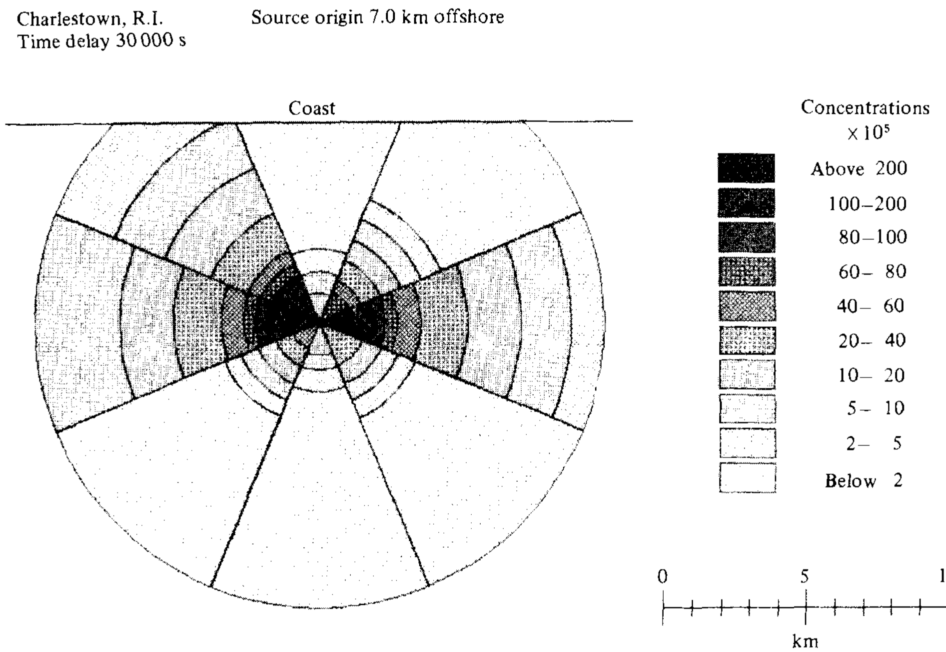

FIGURE 5. Contribution of young puffs to stochastic mean concentration field in example, using current-meter data from Charlestown, R.I., and $t_d = 3 \times 10^4$ s.

Discretized versions of the distributions of interest are now readily seen to be

$$\left.\begin{aligned} \phi(\boldsymbol{u}) &= \frac{f_\beta f_{v\beta}}{v\,\Delta v\,\Delta\beta}, \quad P(\boldsymbol{x}, t_d) = \frac{f_\beta f_{v\beta}(r/t_d)}{r\,\Delta r\,\Delta\beta}, \\ p^{\mathrm{I}}(\boldsymbol{x}) &= \frac{f_\beta}{r\,\Delta\beta} \sum_{v=r/t_d}^{\infty} \frac{f_{v\beta}}{v}, \\ q^{\mathrm{I}}(\boldsymbol{x}) &= \frac{f_\beta}{r\,\Delta\beta} F_\beta\left(\frac{r}{t_d}\right), \quad \gamma^{\mathrm{I}}(\boldsymbol{x}) = b q^{\mathrm{I}}(\boldsymbol{x}). \end{aligned}\right\} \tag{44}$$

In the last of these expressions, b is the diameter of puffs visiting location $\boldsymbol{x}$. For large releases this may simply be taken to be plume diameter at the end of an initial mixing phase.

Using data from a long-term current observation program in Block Island Sound (Snooks & Jacobson 1979), the distribution of extended-source strength $P(\boldsymbol{x}, t_d)$, long-term mean concentration $p^{\mathrm{I}}(\boldsymbol{x})$, and visitation frequency $\gamma^{\mathrm{I}}(\boldsymbol{x})$ has been calculated and is shown in figures 5–7. The duration t_d of the advective phase has been estimated to be 3×10^4 s, or roughly 10 hours, from the observed persistence of currents. Figure 5 shows the concentration $p^{\mathrm{I}}(\boldsymbol{x})$ in units of $10^{-5}\ \mathrm{s\ m^{-2}}$. For the example of §8, 1000 MW waste-heat release, 20 m depth, this has to be multiplied by $12.5\ \mathrm{K\ m^2\ s^{-1}}$ to obtain excess temperature (ignoring heat transfer to the atmosphere). Thus 200 units on the diagram correspond to 0.025 K.

The visitation frequency has been calculated for $b = 300$ m and is shown in figure 6 (as $10^3\gamma^{\mathrm{I}}$), maximum values exceeding 10 %. The extended-source strength $P(x, t_d)$ is given in figure 7 in units of $10^{10}\ \mathrm{m^{-2}}$, so that a value of 100 corresponds to spreading the total source strength over a square 10 km on the side.

All these diagrams show the strong polarization of the currents in the alongshore

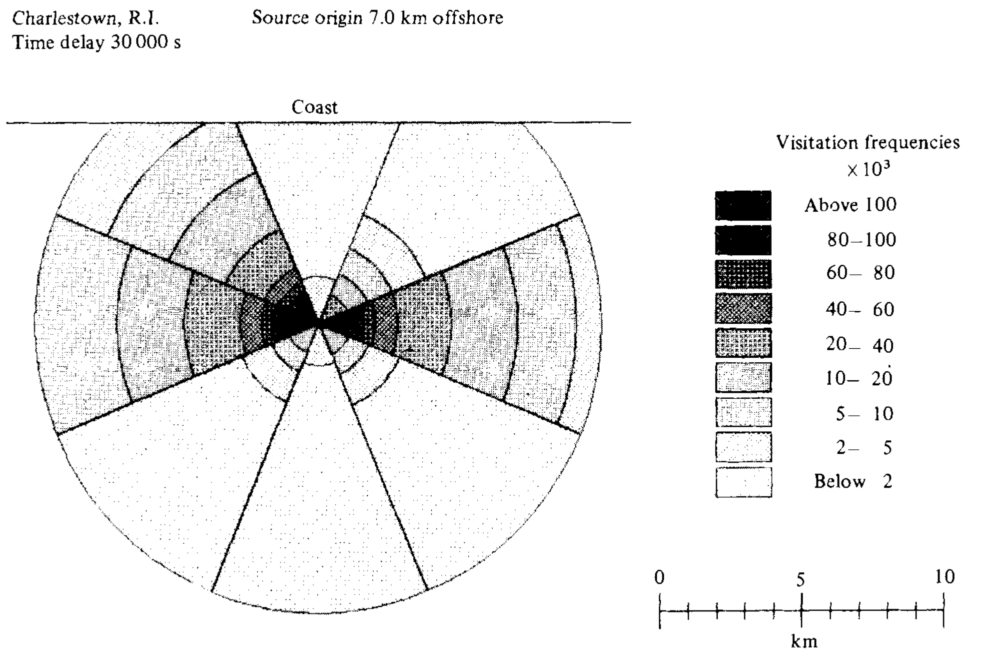

Figure 6. Visitation-frequency distribution in same example as figure 5.

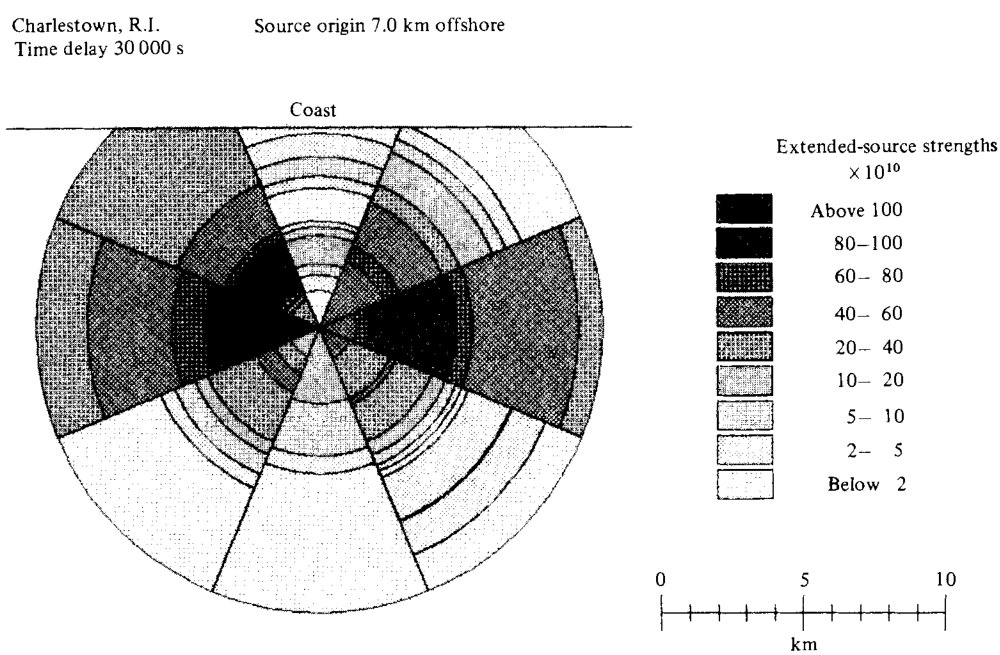

Figure 7. Extended-source intensity distribution determined from current-meter data in same example as figures 5 and 6.

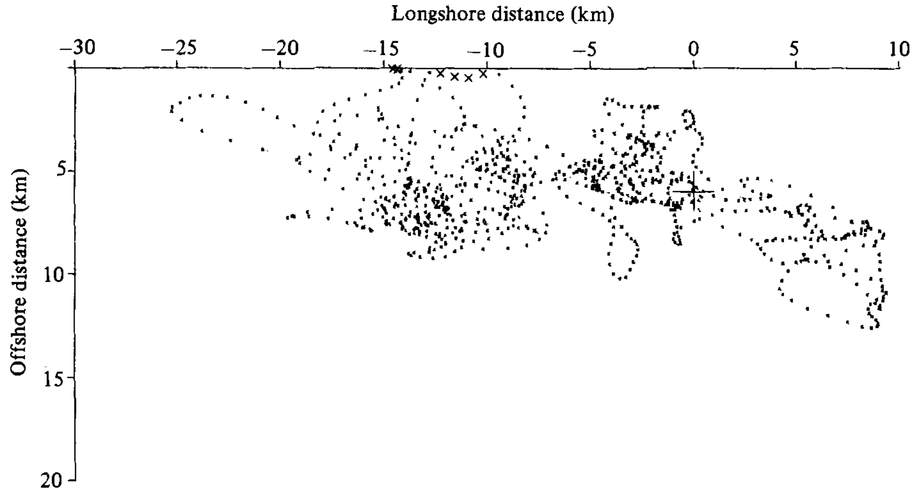

FIGURE 8. Particle displacement distribution 15 h after release, simulated from current-meter data obtained 6 km south of Long Island coastline.

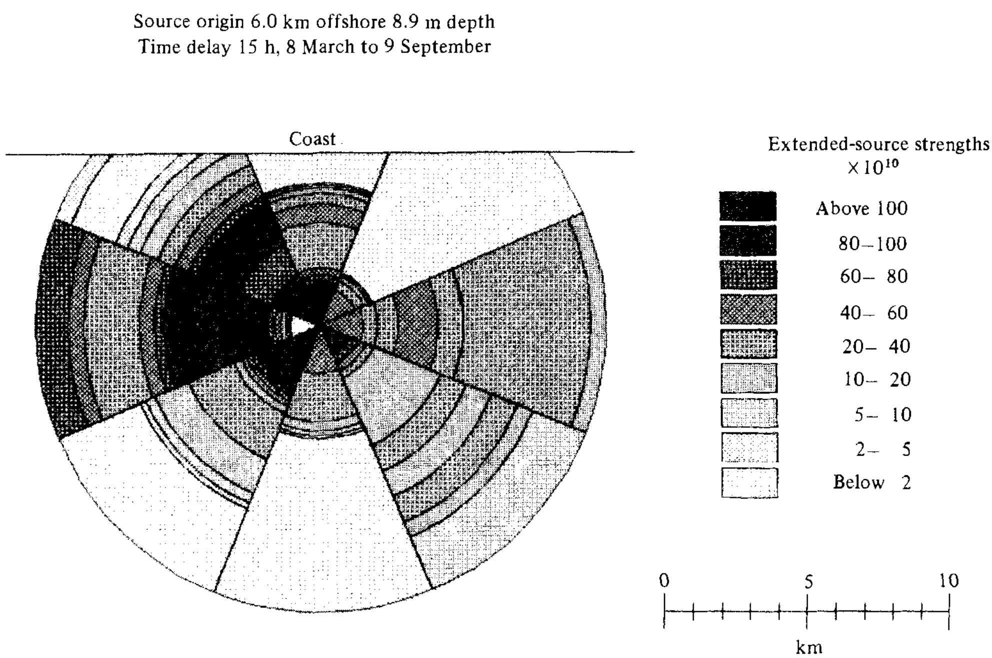

FIGURE 9. Extended-source intensity distribution with $t_d = 15$ h, determined by the same method as in figure 7, for the Long Island example.

direction. In spite of the limited dispersal capacity that would seem to be implied by polarization, low values of $p^{\mathrm{I}}(\boldsymbol{x})$ and a broad distribution $P(\boldsymbol{x}, t_d)$ characterize the net effects of irregular nearshore advection. This is mainly on account of the relatively strong fluctuating currents v which distribute the release over a wide area, best illustrated perhaps by $P(\boldsymbol{x}, t_d)$. Note that no correction has been applied to these diagrams at the coast to conserve mass: where trajectories cross a boundary, the near-source approximation breaks down in any case.

The next order of approximation, commonly applied in air-pollution problems

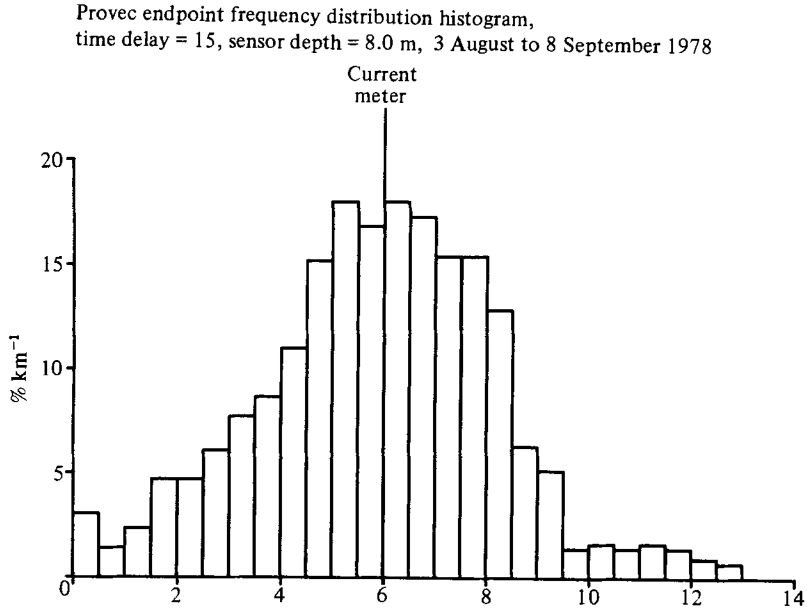

FIGURE 10. Histogram showing density of dots in figure 8 within bands parallel to the coast.

(Pasquill 1974) is to infer Lagrangian displacement from an Eulerian record by the approximation

$$x(t) \approx x_E(t) = \int_0^t u_E(t')\,dt', \tag{45}$$

where $u_E(t)$ is a fixed-point record of velocity. The statistics of $x_E(t)$, derived from a long record, can then be used to infer $P(x, t)$.

Such a calculation has been carried out on some current-meter data obtained off Long Island. Of immediate interest is the extended-source distribution $P(x, t_d)$ for the estimated persistence period of $t_d = 15$ h. Figure 8 shows the distribution of $x(t_d)$ calculated from a 36-day long record, with t_d started every hour. On six occasions the simulated track bumped into the coast: these have been reflected and are shown as larger crosses. For contrast, figure 9 shows the extended-source strength calculated by persistence, from (44). The latter appears to overestimate the size of the extended source somewhat.

Figure 8 shows both dispersion and alongshore advection. The source-extending effect is mainly of consequence in the cross-shore direction. To quantify this, figure 10 has been prepared, showing the frequency distribution of all the dots in figure 8 as a function of offshore distance. The shape of the histogram is clearly well approximated by a Gaussian distribution, with a standard deviation of about 2 km. The standard deviation in the alongshore direction is some 3 times greater. While this is still an approximation, it should give a fairly accurate starting point for the calculation of the background concentration, given the relative insensitivity of the latter to precise source dimensions.

As far as near-source effects are concerned, further insight may, of course, be gained from conventional Gaussian-plume calculations of relative diffusion for short diffusion times. The above discussion should have placed such calculations in perspective and perhaps also clarified the role of models based on the diffusion equation. The latter may be of even more use in more confined geometry, or where nearby sinks have also to be accounted for.

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